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Geometric Phase in a Time-Dependent k-Boson and Fermi System

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Abstract By using the Lewis–Riesenfeld invariant theory, the geometric phase in a time-dependent k-Boson and Fermi system has been studied. It is found that the geometric phase has nothing to do with the field frequency and the coupling coefficient between the Boson and Fermion.

Keywords Geometric phase · k-Boson Fermi system

1 Introduction

It is known that the concept of geometric phase was first introduced by Pancharatnam [1] in studying the interference of classical light in distinct states of polarization. Berry [2] found the quantal counterpart of Pancharatnam's phase in the case of cyclic adiabatic evolution. The extension to non-adiabatic cyclic evolution was developed by Aharonov and Anandan [3]. Samuel et al. [4] generalized the pure state geometric phase by extending it to non-cyclic evolution and sequential projection measurements. The geometric phase is a consequence of quantum kinematics and is thus independent of the detailed nature of the dynamical origin of the path in state space. Mukunda and Simon [5] gave a kinematic approach by taking the path traversed in state space as the primary concept for the geometric phase. Further generalizations and refinements, by relaxing the conditions of adiabaticity, unitarity, and cyclicity of the evolution, have since been carried out [6]. Recently, the geometric phase of the mixed states has also been studied [7–9].

As we known that the quantum invariant theory proposed by Lewis and Riesenfeld [10] is a powerful tool for treating systems with time-dependent Hamiltonians. It was generalized by introducing the concept of basic invariants and used to study the geometric

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phases [11–14] in connection with the exact solutions of the corresponding time-dependent Schrödinger equations. The discovery of Berry's phase is not only a breakthrough in the older theory of quantum adiabatic approximations, but also provides us with new insights in many physical phenomena. The concept of Berry's phase has been developed in some different directions [15–27]. In this letter, by using the Lewis–Riesenfeld invariant theory, we shall study the geometric phase in a time-dependent k-Boson and Fermi system.

2 Model

The Hamiltonian in a time-dependent k-Boson and Fermi system is described as

$$\hat{H} = \omega(t)\hat{a}^\dagger\hat{a} + \Omega(t)\hat{f}^\dagger\hat{f} + \lambda(t)[\hat{a}^k\hat{f}^\dagger + \hat{a}^{\dagger k}\hat{f}], \tag{1}$$

where $\hat{a}(\hat{a}^\dagger)$ are the Bose annihilation (creation) operators which satisfying commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$. $\hat{f}(\hat{f}^\dagger)$ are the Fermi operators and satisfying relations $\{\hat{f}, \hat{f}^\dagger\} = 1$ and $\hat{f}^2 = \hat{f}^{\dagger 2} = 0$. $\omega(t)$ and $\Omega(t)$ are the field frequency, $\lambda(t)$ stands for the coupling coefficient between the Boson and Fermion.

If introducing the following operators $\hat{P}_+ = \hat{a}^k\hat{f}^\dagger$ and $\hat{P}_- = \hat{a}^{\dagger k}\hat{f}$, one has

$$\{\hat{P}_+, \hat{P}_-\} = \hat{a}^k\hat{a}^{\dagger k}\hat{f}^\dagger\hat{f} + \hat{a}^{\dagger k}\hat{a}^k\hat{f}\hat{f}^\dagger, \qquad [\hat{P}_+, \hat{P}_-] = (\hat{a}^k\hat{a}^{\dagger k} + \hat{a}^{\dagger k}\hat{a}^k)\hat{f}^\dagger\hat{f} - \hat{a}^{\dagger k}\hat{a}^k, \tag{2}$$

$$[\hat{P}_+, \hat{a}^\dagger\hat{a}] = k\hat{P}_+, \qquad [\hat{P}_-, \hat{a}^\dagger\hat{a}] = -kP_-, \qquad [\hat{P}_+, \hat{f}^\dagger\hat{f}] = -P_+, \qquad [\hat{P}_-, \hat{f}^\dagger\hat{f}] = P_-. \tag{3}$$

According to the relations

$$\hat{a}^k\hat{a}^{\dagger k} = (\hat{N}+1)(\hat{N}+2)\cdots(\hat{N}+k), \qquad \hat{a}^{\dagger k}\hat{a}^k = \hat{N}(\hat{N}-1)\cdots(\hat{N}-k+1), \tag{4}$$

with $\hat{N} = \hat{a}^\dagger\hat{a}$, one has

$$[\hat{P}_+, \hat{P}_-] = [\hat{F}_k(\hat{N}) + G_k]\hat{f}^\dagger\hat{f} - \hat{W}_k(\hat{N}) - u_k\hat{N}, \tag{5}$$

here $\hat{F}_k(\hat{N})$ and $\hat{W}_k(\hat{N})$ are the functions of operator $\hat{N}$, and $G_k = k!$, $u_k = (-1)^{k-1} \times (k-1)!$. In the following discussion, we consider the stronger coupling case between Boson and Fermion, so we can let $\lambda_{N_a f} = \langle \hat{F}_k(\hat{N})\hat{f}^\dagger\hat{f} - \hat{W}_k(\hat{N})\rangle$, therefore $[\hat{P}_+, \hat{P}_-] = \lambda_{N_a f} + G_k\hat{f}^\dagger\hat{f} - u_k\hat{a}^\dagger\hat{a}$.

Equation (1) becomes

$$\hat{H}(t) = \omega(t)\hat{a}^\dagger\hat{a} + \Omega(t)\hat{f}^\dagger\hat{f} + \lambda(t)[\hat{P}_+ + \hat{P}_-]. \tag{6}$$

It is known that for Fermi vacuum state $|0\rangle$, one has $\hat{f}^\dagger|0\rangle = |1\rangle$, $\hat{f}|0\rangle = 0$, $\hat{f}^\dagger|1\rangle = 0$. If letting

$$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}, \qquad |1\rangle = \begin{pmatrix}0\\1\end{pmatrix}, \tag{7}$$

one has

$$\hat{f} = |0\rangle\langle 1| = \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}, \qquad \hat{f}^\dagger = |1\rangle\langle 0| = \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}, \qquad \hat{f}^\dagger\hat{f}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}, \tag{8}$$

so we can restrict our study in the sub-Hilbert space of the supersymmetric quasi-algebra $\hat{P}_\pm$, $\hat{a}^\dagger\hat{a}$, and $\hat{f}^\dagger\hat{f}$.

3 Dynamical and Geometric Phases

For self-consistent, we first illustrate the Lewis–Riesenfeld (L–R) invariant theory [10]. For a one-dimensional system whose Hamiltonian $\hat{H}(t)$ is time-dependent, then there exists an operator $\hat{I}(t)$ called invariant if it satisfies the equation

$$i\frac{\partial \hat{I}(t)}{\partial t} + [\hat{I}(t), \hat{H}(t)] = 0. \tag{9}$$

The eigenvalue equation of the time-dependent invariant $|\lambda_n, t\rangle$ is given

$$\hat{I}(t)|\lambda_n, t\rangle = \lambda_n|\lambda_n, t\rangle, \tag{10}$$

where $\frac{\partial \lambda_n}{\partial t} = 0$. The time-dependent Schrödinger equation for this system is

$$i\frac{\partial|\psi(t)\rangle_s}{\partial t} = \hat{H}(t)|\psi(t)\rangle_s. \tag{11}$$

According to the L–R invariant theory, the particular solution $|\lambda_n, t\rangle_s$ of (11) is different from the eigenfunction $|\lambda_n, t\rangle$ of $\hat{I}(t)$ only by a phase factor $\exp[i\delta_n(t)]$, i.e.,

$$|\lambda_n, t\rangle_s = \exp[i\delta_n(t)]|\lambda_n, t\rangle, \tag{12}$$

which shows that $|\lambda_n, t\rangle_s$ $(n = 1, 2, \ldots)$ forms a complete set of the solutions of (11). Then the general solution of the Schrödinger equation (11) can be written by

$$|\psi(t)\rangle_s = \sum_n C_n \exp[i\delta_n(t)]|\lambda_n, t\rangle, \tag{13}$$

where

$$\delta_n(t) = \int_0^t dt'\langle\lambda_n, t'|i\frac{\partial}{\partial t'} - \hat{H}(t')|\lambda_n, t'\rangle, \tag{14}$$

and $C_n = \langle\lambda_n, 0|\psi(0)\rangle_s$.

For the system described by Hamiltonian (6), we can define the following invariant

$$\hat{I}(t) = \alpha(t)\hat{P}_+ + \alpha^*(t)\hat{P}_- + \beta\hat{a}^\dagger\hat{a} + \rho\hat{f}^\dagger\hat{f}, \tag{15}$$

here β and ρ are the real numbers and they are independent of time. Substituting (6) and (15) into (9), one has the auxiliary equations

$$i\dot{\alpha}(t) + \alpha(t)[k\omega(t) - \Omega(t)] + \lambda(t)[\rho - k\beta] = 0, \quad \alpha = \alpha^*. \tag{16}$$

where dot denotes the time derivative.

3.1 Dynamical and Geometric Phases when $k =$ odd Number

In order to obtain a time-independent invariant, we can introduce the unitary transformation operator $\hat{V}(t) = \exp[\xi(t)\hat{P}_- - \xi^*(t)\hat{P}_+]$. It is easy to find that when satisfying the following relations

$$\rho + \frac{\alpha(t)G_k[\xi(t) + \xi^*(t)]}{2\sqrt{k!}|\xi(t)|}\sin(2\sqrt{k!}|\xi(t)|) + \frac{(\beta k - \rho)G_k}{2k!}[1 - \cos(2\sqrt{k!}|\xi(t)|)] = 1, \tag{17}$$

$$\beta-\frac{\alpha(t)u_k[\xi(t)+\xi^*(t)]}{2\sqrt{k!}|\xi(t)|}\sin(2\sqrt{k!}|\xi(t)|)+\frac{(\rho-\beta k)u_k}{2k!}[1-\cos(2\sqrt{k!}|\xi(t)|)]=1, \quad (18)$$

$$\frac{\alpha(t)\lambda_{N_af}[\xi(t)+\xi^*(t)]}{2\sqrt{k!}|\xi(t)|}\sin(2\sqrt{k!}|\xi(t)|)+\frac{(\beta k-\rho)\lambda_{N_af}}{2k!}[1-\cos(2\sqrt{k!}|\xi(t)|)]=0, \quad (19)$$

$$\frac{\alpha(t)}{2}[1+\cos(2\sqrt{k!}|\xi(t)|)]-\frac{\alpha(t)\xi^{*2}(t)}{2|\xi(t)|^2}[1-\cos(2\sqrt{k!}|\xi(t)|)] \\ +\frac{(k\beta-\rho)\xi^*(t)}{2\sqrt{k!}|\xi(t)|}\sin(2\sqrt{k!}|\xi(t)|)=0, \quad (20)$$

then a time-independent invariant appears

$$\hat{I}_V\equiv\hat{V}^\dagger(t)\hat{I}(t)\hat{V}(t)=\hat{a}^\dagger\hat{a}+\hat{f}^\dagger\hat{f}. \quad (21)$$

By using the Baker–Campbell–Hausdoff formula [28]

$$\hat{V}^\dagger(t)\frac{\partial\hat{V}(t)}{\partial t}=\frac{\partial\hat{L}}{\partial t}+\frac{1}{2!}\left[\frac{\partial\hat{L}}{\partial t},\hat{L}\right]+\frac{1}{3!}\left[\left[\frac{\partial\hat{L}}{\partial t},\hat{L}\right],\hat{L}\right]+\frac{1}{4!}\left[\left[\left[\frac{\partial\hat{L}}{\partial t},\hat{L}\right],\hat{L}\right],\hat{L}\right]+\cdots, \quad (22)$$

with $\hat{V}(t)=\exp[\hat{L}(t)]$, it is easy to find that when satisfying the following equation

$$\frac{[k\omega(t)-\Omega(t)]\xi^*(t)}{2\sqrt{k!}|\xi(t)|}\sin(2\sqrt{k!}|\xi(t)|)+\frac{\lambda(t)}{2}[1+\cos(2\sqrt{k!}|\xi(t)|)] \\ -\frac{\lambda(t)\xi^{*2}(t)}{2|\xi(t)|^2}[1-\cos(2\sqrt{k!}|\xi(t)|)]+i\xi^*(t) \\ -\frac{i\xi^*[\dot{\xi}(t)\xi^*(t)-\dot{\xi}^*(t)\xi(t)]}{4\sqrt{k!}|\xi(t)|^3}[\sin(2\sqrt{k!}|\xi(t)|)-2\sqrt{k!}|\xi(t)|]=0, \quad (23)$$

one has

$$\hat{H}_V(t)=\hat{V}^\dagger(t)\hat{H}(t)\hat{V}(t)-i\hat{V}^\dagger(t)\frac{\partial\hat{V}(t)}{\partial t} \\ =\omega(t)\hat{a}^\dagger\hat{a}+\Omega(t)\hat{f}^\dagger\hat{f}+\frac{[k\omega(t)-\Omega(t)]G_k}{2k!}[1-\cos(2\sqrt{k!}|\xi(t)|)]\hat{f}^\dagger\hat{f} \\ +\frac{[\Omega(t)-k\omega(t)]u_k}{2k!}[1-\cos(2\sqrt{k!}|\xi(t)|)]\hat{a}^\dagger\hat{a} \\ +\frac{[k\omega(t)-\Omega(t)]\lambda_{N_{af}}}{2k!}[1-\cos(2\sqrt{k!}|\xi(t)|)] \\ +\frac{\lambda(t)\lambda_{N_{af}}[\xi(t)+\xi^*(t)]}{2\sqrt{k!}|\xi(t)|}\sin(2\sqrt{k!}|\xi(t)|) \\ +\frac{\lambda(t)G_k[\xi(t)+\xi^*(t)]}{2\sqrt{k!}|\xi(t)|}\sin(2\sqrt{k!}|\xi(t)|)\hat{f}^\dagger\hat{f} \\ -\frac{\lambda(t)u_k[\xi(t)+\xi^*(t)]}{2\sqrt{k!}|\xi(t)|}\sin(2\sqrt{k!}|\xi(t)|)\hat{a}^\dagger\hat{a}$$

$$- \frac{iG_k[\dot{\xi}(t)\xi^*(t) - \dot{\xi}^*(t)\xi(t)]}{4k!|\xi(t)|^2}[1 - \cos(2\sqrt{k!}|\xi(t)|)]\hat{f}^\dagger \hat{f}$$
$$+ \frac{iu_k[\dot{\xi}(t)\xi^*(t) - \dot{\xi}^*(t)\xi(t)]}{4k!|\xi(t)|^2}[1 - \cos(2\sqrt{k!}|\xi(t)|)]\hat{a}^\dagger \hat{a}$$
$$- \frac{i\lambda_{N_{af}}[\dot{\xi}(t)\xi^*(t) - \dot{\xi}^*(t)\xi(t)]}{4k!|\xi(t)|^2}[1 - \cos(2\sqrt{k!}|\xi(t)|)]. \tag{24}$$

So we can get the particular solution of (11):

$$|\psi(t)\rangle = \exp\left\{-i\int_0^t [\dot{\delta}^d(t') + \dot{\delta}^g(t')]dt'\right\}\hat{V}(t')|n\rangle \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{25}$$

where

$$\begin{aligned}
\dot{\delta}^d(t') &= n\omega(t') + \Omega(t') + \frac{[k\omega(t') - \Omega(t')]G_k}{2k!}[1 - \cos(2\sqrt{k!}|\xi(t')|)] \\
&\quad + \frac{[n\Omega(t') - nk\omega(t')]u_k}{2k!}[1 - \cos(2\sqrt{k!}|\xi(t')|)] \\
&\quad + \frac{[k\omega(t') - \Omega(t')]\lambda_{N_{af}}}{2k!}[1 - \cos(2\sqrt{k!}|\xi(t')|)] \\
&\quad + \frac{\lambda(t')\lambda_{N_{af}}[\xi(t') + \xi^*(t')]}{2\sqrt{k!}|\xi(t')|}\sin(2\sqrt{k!}|\xi(t')|) \\
&\quad + \frac{\lambda(t')G_k[\xi(t') + \xi^*(t')]}{2\sqrt{k!}|\xi(t')|}\sin(2\sqrt{k!}|\xi(t')|) \\
&\quad - \frac{n\lambda(t')u_k[\xi(t') + \xi^*(t')]}{2\sqrt{k!}|\xi(t')|}\sin(2\sqrt{k!}|\xi(t')|),
\end{aligned} \tag{26}$$

$$\begin{aligned}
\dot{\delta}^g(t') &= \frac{i[\dot{\xi}(t')\xi^*(t') - \dot{\xi}^*(t')\xi(t')]}{4k!|\xi(t')|^2} \\
&\quad \times \{nu_k - nu_k\cos(2\sqrt{k!}|\xi(t')|) - (G_k + \lambda_{N_{af}})[1 - \cos(2\sqrt{k!}|\xi(t')|)]\},
\end{aligned} \tag{27}$$

so the geometric phase is

$$\begin{aligned}
\delta^g(t) &= i\int_0^t \frac{[\dot{\xi}(t')\xi^*(t') - \dot{\xi}^*(t')\xi(t')]}{4k!|\xi(t')|^2}[nu_k - nu_k\cos(2\sqrt{k!}|\xi(t')|)]dt' \\
&\quad - i\int_0^t \frac{[\dot{\xi}(t')\xi^*(t') - \dot{\xi}^*(t')\xi(t')]}{4k!|\xi(t')|^2}(G_k + \lambda_{N_{af}})[1 - \cos(2\sqrt{k!}|\xi(t')|)]dt'.
\end{aligned} \tag{28}$$

3.2 Dynamical and Geometric Phases when $k =$ even Number

Similar to Sect. 3.1, we introduce the unitary transformation operator $\hat{V}(t) = \exp[\xi(t)\hat{P}_- - \xi^*(t)\hat{P}_+]$. It is easy to find that when satisfying the following relations

$$\rho + \alpha(t)G_k[\xi(t) + \xi^*(t)] + G_k|\xi(t)|^2[k\beta - \rho] = 1, \tag{29}$$

$$\beta - \alpha(t)u_k[\xi(t) + \xi^*(t)] + u_k|\xi(t)|^2[\rho - k\beta] = 1, \tag{30}$$

$$\alpha(t) + \xi^*(t)[k\beta - \rho] = 0, \qquad \alpha(t)[\xi(t) + \xi^*(t)] + |\xi(t)|^2[k\beta - \rho] = 0, \tag{31}$$

then a time-independent invariant appears

$$\hat{I}_V \equiv \hat{V}^\dagger(t)\hat{I}(t)\hat{V}(t) = \hat{a}^\dagger\hat{a} + \hat{f}^\dagger\hat{f}. \tag{32}$$

When satisfying the relation $[k\omega(t) - \Omega(t)]\xi^*(t) + \lambda(t) + i\dot{\xi}^*(t) = 0$, one has

$$\begin{aligned}\hat{H}_V(t) &= \hat{V}^\dagger(t)\hat{H}(t)\hat{V}(t) - i\hat{V}^\dagger(t)\frac{\partial\hat{V}(t)}{\partial t} \\ &= \omega(t)\hat{a}^\dagger\hat{a} + \Omega(t)\hat{f}^\dagger\hat{f} + [k\omega(t) - \Omega(t)]|\xi(t)|^2 G_k\hat{f}^\dagger\hat{f} \\ &\quad + [\Omega(t) - k\omega(t)]|\xi(t)|^2 u_k\hat{a}^\dagger\hat{a} + \lambda(t)[\xi(t) + \xi^*(t)]\lambda_{N_{af}} \\ &\quad + [k\omega(t) - \Omega(t)]|\xi(t)|^2\lambda_{N_{af}} + \lambda(t)G_k[\xi(t) + \xi^*(t)]\hat{f}^\dagger\hat{f} \\ &\quad - \lambda(t)u_k[\xi(t) + \xi^*(t)]\hat{a}^\dagger\hat{a} - \frac{i}{2}[\dot{\xi}(t)\xi^*(t) - \dot{\xi}^*(t)\xi(t)][\lambda_{N_{af}} + G_k\hat{f}^\dagger\hat{f} - u_k\hat{a}^\dagger\hat{a}].\end{aligned} \tag{33}$$

So we can get the particular solution of (11):

$$|\psi(t)\rangle = \exp\left\{-i\int_0^t[\dot{\delta}^d(t') + \dot{\delta}^g(t')]dt'\right\}\hat{V}(t')|n\rangle \otimes \begin{pmatrix}0\\1\end{pmatrix}, \tag{34}$$

where

$$\begin{aligned}\dot{\delta}^d(t') &= n\omega(t') + \Omega(t') + [k\omega(t') - \Omega(t')]|\xi(t')|^2 G_k \\ &\quad + n[\Omega(t') - k\omega(t')]|\xi(t')|^2 u_k + \lambda(t')[\xi(t') + \xi^*(t')]\lambda_{N_{af}} \\ &\quad + [k\omega(t') - \Omega(t')]|\xi(t')|^2\lambda_{N_{af}} + \lambda(t')G_k[\xi(t') + \xi^*(t')] \\ &\quad - n\lambda(t')u_k[\xi(t') + \xi^*(t')],\end{aligned} \tag{35}$$

$$\dot{\delta}^g(t') = -\frac{i}{2}[\dot{\xi}(t')\xi^*(t') - \dot{\xi}^*(t')\xi(t')][\lambda_{N_{af}} + G_k - nu_k], \tag{36}$$

so the geometric phase is

$$\delta^g(t) = \int_0^t\left\{\frac{i}{2}[\dot{\xi}(t')\xi^*(t') - \dot{\xi}^*(t')\xi(t')][\lambda_{N_{af}} + G_k - nu_k]\right\}dt'. \tag{37}$$

According to Sects. 3.1 and 3.2, it is found that the geometric phase has nothing to do with the field frequency and the coupling coefficient between the Boson and Fermion.

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